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Discrete Mathematics 306 (2006) 552–563
www.elsevier.com/locate/disc

Restricted signed permutations counted by the Schröder numbers

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Received 1 September 2005; received in revised form 8 December 2005; accepted 25 January 2006

Abstract

Gire, West, and Kremer have found ten classes of restricted permutations counted by the large Schröder numbers, no two of which are trivially Wilf-equivalent. In this paper we enumerate eleven classes of restricted signed permutations counted by the large Schröder numbers, no two of which are trivially Wilf-equivalent. We obtain five of these enumerations by elementary methods, five by displaying isomorphisms with the classical Schröder generating tree, and one by giving an isomorphism with a new Schröder generating tree. When combined with a result of Egge and a computer search, this completes the classification of restricted signed permutations counted by the large Schröder numbers in which the set of restrictions consists of two patterns of length 2 and two of length 3.
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Keywords: Restricted permutation; Pattern-avoiding permutation; Forbidden subsequence; Schröder number; Signed permutation; Generating tree

1. Introduction and notation

Let B_n denote the set of permutations of $\{1, 2, \ldots, n\}$, written in one-line notation, in which each element may or may not have a bar above it. We refer to the elements of B_n as *signed permutations*. We write S_n to denote the set of elements of B_n with no bars, and we refer to these elements as *classical permutations*. For any signed permutation π we write $|\pi|$ to denote the length of π and we write $\pi(i)$ to denote the ith entry of π.

Suppose π and σ are signed permutations. We say a subsequence of π has *type* σ whenever it has all of the same pairwise comparisons as σ and an entry in the subsequence of π is barred if and only if the corresponding entry in σ is barred. For example, the subsequence $3\bar{4}71$ of the signed permutation $93\bar{4}72\bar{8}\bar{5}16$ has type $2\bar{3}41$. We say π *avoids* σ whenever π has no subsequence of type σ. For example, the signed permutation $93\bar{4}72\bar{8}\bar{5}16$ avoids $\bar{3}2\bar{1}$ and 1432 but it has $92\bar{5}$ as a subsequence so it does not avoid $31\bar{2}$. In this setting σ is sometimes called a *pattern* or a *forbidden subsequence* and π is sometimes called a *restricted permutation* or a *pattern-avoiding permutation*. In this paper we will be interested in signed permutations which avoid several patterns, so for any set R of signed permutations we write $B_n(R)$ to denote the set of signed permutations of length n which avoid every pattern in R and we write $B(R)$ to denote the set of all signed permutations which avoid every pattern in R. When $R = \{\pi_1, \ldots, \pi_r\}$ we often write $B_n(R) = B_n(\pi_1, \ldots, \pi_r)$ and $B(R) = B(\pi_1, \ldots, \pi_r)$. When we wish to discuss classical permutations, we replace B with S in the above notation.

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0012-365X/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.disc.2006.01.013

Suppose R_1 and R_2 are sets of signed permutations. We say R_1 and R_2 are *Wilf-equivalent* whenever $|B_n(R_1)| = |B_n(R_2)|$ for all $n \geqslant 0$. There are four natural operations which preserve Wilf-equivalence classes:

- the bar operator, which replaces each barred entry in a signed permutation with its unbarred counterpart and vice versa;
- the reverse operator, which writes the entries of a signed permutation in reverse order;
- the complement operator, which replaces each entry $\pi(i)$ of a signed permutation π with $|\pi| + 1 - \pi(i)$;
- the inverse operator, which replaces each signed permutation with its group-theoretic inverse.

For example, if $\pi = 2\bar{4}13$ then $\mathrm{bar}(\pi) = \bar{2}4\overline{13}$, $\mathrm{reverse}(\pi) = 31\bar{4}2$, $\mathrm{complement}(\pi) = 3\bar{1}42$, and $\mathrm{inverse}(\pi) = 314\bar{2}$. If we write each signed permutation as a square permutation matrix in which barred entries are represented by -1s, then the bar operator is multiplication by -1, the reverse operator is the reflection over the vertical axis, the complement operator is the reflection over the horizontal axis, and the inverse operator is both the reflection over the main diagonal and the usual matrix inverse. From this one can show that the group G generated by these operations is isomorphic to $D_8 \oplus Z_2$, where D_8 is the dihedral group of order 8. We say R_1 is *trivially Wilf-equivalent* to R_2 whenever R_2 is the image of R_1 under some element of G.

The focus of this paper is on restricted signed permutations, but it includes a major role for the large Schröder numbers. (There are also small Schröder numbers, which are, up to a shift in index, half of the large Schröder numbers.) The *large Schröder numbers* (hereafter just the Schröder numbers) may be recursively defined by $r_0 = 1$ and $r_n = r_{n-1} + \sum_{k=1}^{n} r_{k-1} r_{n-k}$ for $n \geqslant 0$. From this definition one can show that the generating function for the Schröder numbers is given by

$$\sum_{n=0}^{\infty} r_n x^n = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}. \tag{1}$$

The Schröder numbers are closely related to the ubiquitous Catalan numbers, and in particular one can also show that

$$r_n = \sum_{d=0}^{n} \binom{2n-d}{d} C_{n-d} \quad (n \geqslant 0). \tag{2}$$

Here C_n is the nth Catalan number, which may be defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$. For a list of some of the known combinatorial interpretations of the Schröder numbers, see [12, pp. 239–240].

Gire [7], West [13], and Kremer [8] have found ten classes of classical pattern-avoiding permutations which are enumerated by the Schröder numbers. (See [3] for other work along these lines.) Each of the corresponding sets of forbidden patterns consists of two classical permutations, each of length 4, and no two of these sets are trivially Wilf-equivalent. A computer search reveals that every pair of classical permutations of length 4 whose pattern-avoiding permutations are enumerated by Schröder numbers is trivially Wilf-equivalent to one of the ten sets found by Gire, West, and Kremer, but to the best of our knowledge it is not known whether there are other classes of classical pattern-avoiding permutations which are enumerated by the Schröder numbers.

The enumeration of restricted signed permutations was first considered by Simion [11] and studied further by Mansour and West [9]. In this paper we continue this study, considering in particular the following twelve sets of forbidden signed permutations

$$T_1 = \{2\bar{1}, \overline{21}, \bar{3}12, 312\}, \quad T_5 = \{\overline{21}, \bar{2}1, 1\bar{2}3, 312\}, \quad T_9 = \{21, 2\bar{1}, \overline{321}, \bar{3}1\bar{2}\},$$

$$T_2 = \{\overline{21}, \bar{2}1, 1\bar{2}3, 123\}, \quad T_6 = \{\overline{21}, \bar{2}1, 1\bar{2}3, 321\}, \quad T_{10} = \{21, 2\bar{1}, \overline{231}, \bar{3}1\bar{2}\},$$

$$T_3 = \{\overline{21}, \bar{2}1, 1\bar{2}3, 132\}, \quad T_7 = \{21, 2\bar{1}, \overline{312}, \bar{3}1\bar{2}\}, \quad T_{11} = \{21, 2\bar{1}, \overline{132}, 1\overline{32}\},$$

$$T_4 = \{\overline{21}, \bar{2}1, 1\bar{2}3, 231\}, \quad T_8 = \{21, 2\bar{1}, \overline{321}, \overline{321}\}, \quad T_{12} = \{21, \bar{2}1, \overline{321}, \overline{312}\}.$$

Egge has previously shown [6] that $|B_n(T_1)| = r_n$ for all $n \geqslant 0$. Here we begin by using elementary methods to prove that if T is any set of classical permutations then $|B_n(\overline{21}, \bar{2}1, 1\bar{2}3, T)|$ is a convolution of $|S_n(T)|$ with a certain sequence of binomial coefficients. Combining this with (2), we conclude that for any set R among T_2–T_6 we have $|B_n(R)| = r_n$

for all $n \geqslant 0$. Next we recall some basic facts concerning generating trees, including the classical Schröder tree, and we describe how each set of restricted signed permutations can be organized into a generating tree in a natural way. For each set R among T_7–T_{11} we give an isomorphism between the generating tree for $B(R)$ and the classical Schröder tree. It follows that for each set R among T_7–T_{11} we have $|B_n(R)| = r_n$ for all $n \geqslant 0$. We then introduce a new generating tree, which we call the tilted Schröder tree. We use the kernel method to prove that for all $n \geqslant 0$ this tree has exactly r_n nodes on level n, and we give an isomorphism between the generating tree for $B(T_{12})$ and the tilted Schröder tree. It follows that $|B_n(T_{12})| = r_n$ for all $n \geqslant 0$.

It is routine to check that no two of the sets of forbidden signed permutations in T_1–T_{12} are trivially Wilf-equivalent. (As an aside, T_3, T_7, T_9, T_{10}, and T_{11} are each trivially Wilf-equivalent to 15 other sets, and the remaining T_i are each trivially Wilf-equivalent to 7 other sets.) A computer search reveals that if R is any set of signed permutations consisting of two signed permutations of length 2 and two of length 3, and $|B_n(R)| = r_n$ for $0 \leqslant n \leqslant 5$, then R is trivially Wilf-equivalent to one of T_1–T_{12}. In short, up to trivial Wilf-equivalence, this paper completes the classification of restricted signed permutations counted by the Schröder numbers whose forbidden patterns consist of two patterns of length 2 and two of length 3. To the best of our knowledge, it is not known whether there are other classes of restricted signed permutations counted by the Schröder numbers.

2. A convolution with certain binomial coefficients

Some enumerations of pattern-avoiding signed permutations can be obtained from corresponding enumerations of classical pattern-avoiding permutations. For instance, the fact that $|B_n| = 2^n|S_n|$ for $n \geqslant 0$ can be generalized as follows.

Definition 2.1. For any set T of classical permutations, we write $\hat{T}$ to denote the set of signed permutations obtained by putting bars over the entries in the elements of T in all possible ways.

To illustrate Definition 2.1, we observe that if $T = \{12\}$ then $\hat{T} = \{12, \bar{1}2, 1\bar{2}, \bar{1}\bar{2}\}$.

Proposition 2.2. *For any set T of classical permutations, we have*

$$B_n(\hat{T}) = \widehat{S_n(T)} \quad (n \geqslant 0). \tag{3}$$

In particular,

$$|B_n(\hat{T})| = 2^n |S_n(T)| \quad (n \geqslant 0). \tag{4}$$

Proof. To prove (3), note that $\pi \in B_n(\hat{T})$ if and only if $\|\pi\| \in S_n(T)$, where $\|\pi\|$ is the classical permutation obtained by removing all bars from π.

Line (4) is immediate from (3). $\square$

Proposition 2.2 allows us to recover a result of Mansour and West.

Corollary 2.3 (*Mansour and West [9, Eq. (4.4)]*). *For all $n \geqslant 0$ we have* $|B_n(12, \bar{1}2, 1\bar{2}, \bar{1}\bar{2})| = 2^n$.

Proof. Set $T = \{12\}$ in (4) and use the fact that $|S_n(12)| = 1$ for all $n \geqslant 0$. $\square$

The main result of this section relates enumerations of classical pattern-avoiding permutations with pattern-avoiding signed permutations by a convolution with certain binomial coefficients. To prove the result, we will use the following technical lemma.

Lemma 2.4. *A signed permutation π avoids $\bar{2}\bar{1}$, $\bar{2}1$, and $1\bar{2}3$ if and only if both of the following hold.*

(i) *The barred entries of π are in increasing order.*
(ii) *If a barred entry of π has an unbarred entry to its right, then the barred entry is less than all unbarred entries of π.*

Proof. ($\Longrightarrow$) Suppose π avoids $\overline{21}$, $\bar{2}1$, and $1\bar{2}3$. First observe that (i) follows from the fact that π avoids $\overline{21}$. To prove (ii), suppose $\pi(i)$ is barred, $\pi(j)$ is unbarred, and $i < j$. If there is an unbarred entry a of π to the right of $\pi(i)$ such that $\pi(i) > a$ then $\pi(i)a$ has type $\bar{2}1$. If there is an unbarred entry a of π to the left of $\pi(i)$ such that $\pi(i) > a$ then $a\pi(i)\pi(j)$ has type $1\bar{2}3$. Now (ii) follows.

($\Longleftarrow$) Suppose (i) and (ii) hold for a signed permutation π. By (i), π avoids $\overline{21}$. If π contains a pattern of type $\bar{2}1$ or $1\bar{2}3$ then the entry playing the role of the $\bar{2}$ violates (ii), so π avoids $\bar{2}1$ and $1\bar{2}3$. $\square$

In our next result, we use Lemma 2.4 to count signed permutations which have a given number of barred entries and which avoid $\overline{21}$, $\bar{2}1$, $1\bar{2}3$, and any set T of classical patterns.

Proposition 2.5. *Let T denote a set of classical permutations. Then for all $n \geqslant 0$ and all d such that $0 \leqslant d \leqslant n$ there are $\binom{2n-d}{d}|S_{n-d}(T)|$ signed permutations in B_n which avoid $\overline{21}$, $\bar{2}1$, $1\bar{2}3$, and T and which have exactly d barred entries.*

Proof. Fix $n \geqslant 0$ and d such that $0 \leqslant d \leqslant n$. We first describe an algorithm for constructing a signed permutation π which avoids $\overline{21}$, $\bar{2}1$, $1\bar{2}3$, and T and which has exactly d barred entries.

1. Choose a classical permutation $\sigma \in S_{n-d}(T)$.
2. Construct the graph of σ by placing (unbarred) dots at the points $(i, \sigma(i))$ for $1 \leqslant i \leqslant n-d$.
3. View the $n-d$ vertical lines $x=\frac{1}{2}, x=\frac{3}{2}, \ldots, x=n-d-\frac{1}{2}$ and the $n-d+1$ horizontal lines $y=\frac{1}{2}, y=\frac{3}{2}, \ldots, y=n-d+\frac{1}{2}$ as baskets, and place a total of d indistinguishable balls in the $2n-2d+1$ baskets. That is, distribute d barred dots among the points $\left(\frac{1}{2}, 0\right), \left(\frac{3}{2}, 0\right), \ldots, \left(n-d-\frac{1}{2}, 0\right), \left(n-d+\frac{1}{2}, \frac{1}{2}\right), \left(n-d+\frac{1}{2}, \frac{3}{2}\right), \ldots, (n-d+\frac{1}{2}, n-d+\frac{1}{2})$, allowing multiple dots at each point.
4. Produce the graph of a signed permutation as follows.
   (a) Space the points of the graph of σ and the inserted dots in order horizontally, starting with the dots on $x=\frac{1}{2}$, followed by the dot at $(1, \sigma(1))$, followed by the dots on $x=\frac{3}{2}$, etc., followed by the dot at $(n-d, \sigma(n-d))$, followed by the dots on $y=\frac{1}{2}$, followed by the dots on $y=\frac{3}{2}$, etc.
   As an example, suppose $\sigma = 231$, there are two dots on $x=\frac{1}{2}$, there is one dot on $x=\frac{3}{2}$, and there is one dot on $y=\frac{3}{2}$. After this step there will be dots on (1, 0), (2, 0), (3, 2), (4, 0), (5, 3), (6, 1), and $\left(7, \frac{3}{2}\right)$. Of these, only the dots on (3, 2), (5, 3), and (6, 1) will be unbarred.
   (b) Space the points of the graph of σ and the inserted dots in order vertically, starting with the dots which were on $x=\frac{1}{2}$, followed by the dots which were on $x=\frac{3}{2}$, etc., followed by the dots which were on $x=n-d-\frac{1}{2}$, followed by the dots on $y=\frac{1}{2}$, followed by the dot which was at $(\sigma^{-1}(1), 1)$, followed by the dots on $y=\frac{3}{2}$, followed by the dot which was at $(\sigma^{-1}(2), 2)$, etc.
   In the example begun in (a), the resulting diagram will have barred dots at (1, 1), (2, 2), (4, 3), and (7, 5), and it will have unbarred dots at (6, 4), (3, 6), and (5, 7). The associated signed permutation is $\bar{1}\bar{2}6\bar{3}74\bar{5}$.

Observe that step 4 guarantees that this algorithm always produces (the graph of) a signed permutation of length n with exactly d barred entries. In view of Lemma 2.4, the algorithm produces only signed permutations which avoid $\overline{21}$, $\bar{2}1$, $1\bar{2}3$, and T, and each such signed permutation is produced in exactly one way. Steps 2 and 4 may each be performed in just one way, and step 1 may be performed in $|S_{n-d}(T)|$ ways. In step 3 we are inserting d indistinguishable balls in $2n-2d+1$ distinguishable baskets; there are $\binom{2n-d}{d}$ ways to do this. The result follows. $\square$

Proposition 2.5 gives us two sets of signed permutations counted by the Fibonacci numbers, one of which was previously found by Mansour and West [9, Eq. (3.5)] and later by Egge [6].

Proposition 2.6. *For any permutation $\sigma \in S_2$ we have*

$$|B_n(\overline{21}, \bar{2}1, 1\bar{2}3, \sigma)| = F_{2n+1} \quad (n \geqslant 0), \tag{5}$$

where F_n is the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$.

Proof. Set $T = \{\sigma\}$ in Proposition 2.5 and use the fact that $|S_n(T)| = 1$ for all $n \geqslant 0$ to obtain

$$|B_n(\overline{21}, \overline{2}1, 1\overline{2}3, \sigma)| = \sum_{d=0}^{n} \binom{2n-d}{d} \quad (n \geqslant 0).$$

This last expression is well-known to be equal to F_{2n+1}. (See [4, Theorem 7.1.2], for instance.) □

Proposition 2.5 also gives us several sets of signed permutations counted by the Schröder numbers.

Theorem 2.7. *For any permutation $\sigma \in S_3$ we have*

$$|B_n(\overline{21}, \overline{2}1, 1\overline{2}3, \sigma)| = r_n \quad (n \geqslant 0). \tag{6}$$

We observe that the sets corresponding to $\sigma = 132$ and $\sigma = 213$ are trivially Wilf-equivalent.

Proof. Set $T = \{\sigma\}$ in Proposition 2.5, use the fact that $|S_n(\sigma)| = C_n$ for $n \geqslant 0$, and use (2) to simplify the result. □

Observe that by choosing σ appropriately in Theorem 2.7 we obtain the five sets T_2–T_6 of forbidden patterns given in the Introduction. The sixth choice of σ is $\sigma = 213$, but the resulting set $\{\overline{21}, \overline{2}1, 1\overline{2}3, 213\}$ can be transformed into T_3 by first applying the reverse map, then the complement map, and finally the inverse map.

3. An interlude on generating trees

For our purposes, a *generating tree* is a rooted, labeled tree in which the label of each node determines the node's number of children and their labels. In many generating trees the label of each node is its number of children, but this is not required. We generally specify a particular generating tree by giving two pieces of data:

- the label of the root node;
- a list of succession rules, which state for each label k the number of children a node with label k has and what their labels are.

Many generating trees of combinatorial significance are known; [1,14] contain several examples of interest. We content ourselves here by recalling one which is particularly relevant.

Example 3.1 (*West [14, Ex. 5]*). The *classical Schröder generating tree* is given by

- Root: (2);
- Rule: $(k) \to (3)(4)\cdots(k-1)(k)(k+1)(k+1)$ for $k \geqslant 2$.

In particular, $(2) \to (3)(3)$ and $(3) \to (3)(4)(4)$.

Given a generating tree, we are often interested in how many nodes it has on level n, where the root is on level 0. The classical Schröder tree gets its name from the well-known fact that it has r_n nodes on level n for all $n \geqslant 0$.

Signed pattern-avoiding permutations (as well as classical pattern-avoiding permutations) have a natural generating tree structure. To describe this structure, suppose T is a set of forbidden signed permutations. The nodes on level n of the associated generating tree are the elements of $B_n(T)$, and in the absence of a simpler labeling scheme, we regard each node as being labeled with its associated signed permutation. Each signed permutation $\pi \in B_{n-1}(T)$ has n spaces in which an n or an $\overline{n}$ may be inserted to produce a signed permutation in B_n, but in general only some of the resulting signed permutations will avoid T. For us the children of $\pi \in B_{n-1}(T)$ are those signed permutations which are obtained from π by inserting n or $\overline{n}$ and which avoid T. Given $\pi \in B_{n-1}(T)$, we call an insertion space *unbar-active* (resp. *bar-active*) whenever insertion of n (resp. $\overline{n}$) in that space produces an element of $B_n(T)$ and we call the space *unbar-inactive* (resp. *bar-inactive*) otherwise.

Example 3.2. Suppose $T=\{\overline{2}1, 123\}$ and $\pi=4\overline{1}52\overline{3}$. Then the left-most three spaces of π are unbar-active, the remaining spaces are unbar-inactive, the right-most two spaces of π are bar-active, and the remaining spaces are bar-inactive.

Suppose T is a set of forbidden patterns and $\pi \in B_{n-1}(T)$ for some $n \geqslant 1$. We note that if a space is unbar-inactive in π then it retains that status in all of the children of π. Inserting n into an unbar-inactive space of π produces a forbidden pattern, and since π avoids T, the newly inserted entry n must participate in that forbidden pattern. Along the same lines, suppose spaces s_1 and s_2 in π are unbar-active, but insertion of n in s_1 causes s_2 to become unbar-inactive. Then the signed permutation obtained from π by inserting n in s_1 and $n+1$ in s_2 contains a forbidden pattern, and n and $n+1$ both participate in that pattern. Similar comments hold for bar-active and bar-inactive spaces.

4. Isomorphisms to the classical Schröder tree

In this section we consider the following five sets of forbidden signed permutations.

- $T_7 = \{21, 2\overline{1}, \overline{312}, \overline{3}1\overline{2}\}$
- $T_8 = \{21, 2\overline{1}, \overline{321}, \overline{32}1\}$
- $T_9 = \{21, 2\overline{1}, \overline{321}, \overline{3}1\overline{2}\}$
- $T_{10} = \{21, 2\overline{1}, \overline{231}, \overline{3}1\overline{2}\}$
- $T_{11} = \{21, 2\overline{1}, \overline{132}, 1\overline{32}\}$

Following [8,13], for each of T_7–T_{11} we give an isomorphism between the generating tree for the associated pattern-avoiding signed permutations and the classical Schröder generating tree. (For another approach which can be used to enumerate these signed permutations, see [2].) To obtain our isomorphisms, we analyze the behavior of the unbar-active and bar-active spaces of a signed permutation when n or $\overline{n}$ is inserted. We begin by observing that if π avoids any of T_7–T_{11} then just one space in π can be unbar-active.

Lemma 4.1. *Suppose $n \geqslant 2$ and T is one of T_7–T_{11} above. Fix $\pi \in B_{n-1}(T)$. Then the right-most space of π is unbar-active and all other spaces of π are unbar-inactive.*

Proof. Observe that no forbidden pattern ends with its largest entry, so the right-most space of π is unbar-active. However, inserting n into any other space will produce a subsequence of type 21 or a subsequence of type $2\overline{1}$, depending on whether the right-most entry of π is barred. Both patterns are forbidden, so no other space in π is unbar-active. $\square$

Next we analyze the effect inserting n has on the bar-active spaces.

Lemma 4.2. *Suppose $n \geqslant 2$ and T is one of T_7–T_{11} above. Fix $\pi \in B_{n-1}(T)$ and let π^+ denote the signed permutation obtained by appending n to the right end of π. Then the following hold.*

(i) *The right-most two spaces of π^+ are bar-active.*
(ii) *Suppose s is a space in π^+ which is not one of the right-most two spaces. Abusing notation, we identify s with the corresponding space in π. Then s is bar-active in π^+ if and only if it is bar-active in π.*

Proof. (i) None of the forbidden patterns of length 2 contains $\overline{2}$ and none of the forbidden patterns of length 3 ends with $\overline{3}$, so the right-most space of π^+ is bar-active. Similarly, none of the forbidden patterns of length 3 contains $\overline{3}2$, so the second space from the right end of π^+ is bar-active.

(ii) If s is bar-inactive in π then it remains so in π^+, so suppose by way of contradiction that s is bar-active in π and bar-inactive in π^+. Then inserting $\overline{n+1}$ in π^+ at s produces a forbidden pattern in which both n and $\overline{n+1}$ participate.

But none of the forbidden patterns of length 2 contains $\overline{2}$ and none of the forbidden patterns of length 3 contains 2. This is a contradiction, so s must be bar-active in π^+. $\square$

We now give an isomorphism between the generating tree for $B(T_7)$ and the classical Schröder generating tree. This example illustrates one of the simpler ways the classical Schröder tree can appear as a tree of pattern-avoiding permutations.

Theorem 4.3. *For each signed permutation $\pi \in B(T_7)$, let $f_7(\pi)$ denote one plus the number of spaces in π with no subsequence of type $\overline{12}$ or $1\overline{2}$ to their right. Then the map $\pi \mapsto f_7(\pi)$ is an isomorphism of generating trees between the generating tree for $B(T_7)$ and the classical Schröder generating tree. In particular,*

$$|B_n(21, 2\overline{1}, \overline{312}, \overline{3}1\overline{2})| = r_n \quad (n \geqslant 0).$$

Proof. To begin, observe that $f_7(\emptyset) = 2$ and $f_7(1) = f_7(\overline{1}) = 3$, so both trees have the same first two levels, and it is sufficient to show they have the same succession rules.

Fix $n \geqslant 2$ and suppose $\pi \in B_{n-1}(T_7)$. Observe that since $\overline{2}$ does not appear in the forbidden patterns of length 2, and since the other two forbidden patterns are $\overline{312}$ and $\overline{3}1\overline{2}$, a space in π is bar-active if and only if it has no subsequence of type $\overline{12}$ or $1\overline{2}$ to its right. In view of Lemma 4.1, the signed permutation π has $f_7(\pi) - 1$ spaces which are bar-active and 1 space which is unbar-active, so π has $f_7(\pi)$ children. In view of Lemma 4.2, the child of π obtained by inserting n has $f_7(\pi) + 1$ children, so its image under f_7 is $f_7(\pi) + 1$. To count the children of the signed permutations obtained by inserting $\overline{n}$ into π, observe that the inserted $\overline{n}$ and the entry immediately to its left form a subsequence of type $\overline{12}$ or $1\overline{2}$. Therefore the bar-active spaces in the new signed permutation are the spaces to the right of $\overline{n}$ and the space immediately to the left of $\overline{n}$. Hence the children of π have $3, 4, \ldots, f_7(\pi), f_7(\pi) + 1, f_7(\pi) + 1$ children, and the result follows. $\square$

Next we describe an isomorphism between the generating tree for $B(T_8)$ and the classical Schröder generating tree. Here the classical Schröder tree appears in a slightly more complicated way.

Theorem 4.4. *For each signed permutation $\pi \in B(T_8)$, let $f_8(\pi)$ denote one plus the number of spaces in π with no subsequence of type $\overline{21}$ or $\overline{2}1$ to their right. Then the map $\pi \mapsto f_8(\pi)$ is an isomorphism of generating trees between the generating tree for $B(T_8)$ and the classical Schröder generating tree. In particular,*

$$|B_n(21, 2\overline{1}, \overline{321}, \overline{32}1)| = r_n \quad (n \geqslant 0).$$

Proof. To begin, observe that $f_8(\emptyset) = 2$ and $f_8(1) = f_8(\overline{1}) = 3$, so both trees have the same first two levels, and it is sufficient to show they have the same succession rules.

Fix $n \geqslant 2$ and suppose $\pi \in B_{n-1}(T_8)$. Observe that since $\overline{2}$ does not appear in the forbidden patterns of length 2, and since the other two forbidden patterns are $\overline{321}$ and $\overline{32}1$, a space in π is bar-active if and only if it has no subsequence of type $\overline{21}$ or $\overline{2}1$ to its right. In view of Lemma 4.1, the signed permutation π has $f_8(\pi) - 1$ spaces which are bar-active and 1 space which is unbar-active, so π has $f_8(\pi)$ children. In view of Lemma 4.2, the child of π obtained by inserting n has $f_8(\pi) + 1$ children, so its image under f_8 is $f_8(\pi) + 1$. To count the children of the signed permutations obtained by inserting $\overline{n}$ into π, first observe that if $\overline{n}$ is inserted in the right-most space, all bar-active spaces remain bar-active. If $\overline{n}$ is inserted in any other space, it forms a subsequence of type $\overline{21}$ or $\overline{2}1$with the entry immediately to its right. In this case all spaces to the left of $\overline{n}$ become bar-inactive, while those to the right of $\overline{n}$ remain bar-active. Therefore the children of π have $f_8(\pi) + 1, 3, 4, \ldots, f_8(\pi), f_8(\pi) + 1$ children, and the result follows. $\square$

The generating tree for $B(T_9)$ is the classical Schröder tree in an even more complicated way.

Theorem 4.5. *For each signed permutation $\pi \in B(T_9)$, let $f_9(\pi)$ denote one plus the number of spaces in π with no subsequence of type $\overline{21}$ or $1\overline{2}$ to their right. Then the map $\pi \mapsto f_9(\pi)$ is an isomorphism of generating trees between the generating tree for $B(T_9)$ and the classical Schröder generating tree. In particular,*

$$|B_n(21, 2\overline{1}, \overline{321}, \overline{3}1\overline{2})| = r_n \quad (n \geqslant 0).$$

Proof. To begin, observe that $f_9(\emptyset)=2$, and $f_9(1)=f_9(\overline{1})=3$, so both trees have the same first two levels, and it is sufficient to show they have the same succession rules.

Fix $n\geqslant 2$ and suppose $\pi\in B_{n-1}(T_9)$. Observe that since $\overline{2}$ does not appear in the forbidden patterns of length 2, and since the other two forbidden patterns are $\overline{321}$ and $\overline{3}1\overline{2}$, a space in π is bar-active if and only if it has no subsequence of type $\overline{21}$ or $1\overline{2}$ to its right. In view of Lemma 4.1, the signed permutation π has $f_9(\pi)-1$ spaces which are bar-active and 1 space which is unbar-active, so π has $f_9(\pi)$ children. In view of Lemma 4.2, the child of π obtained by inserting n has $f_9(\pi)+1$ children, so its image under f_9 is $f_9(\pi)+1$. To count the children of the signed permutations obtained by inserting $\overline{n}$ into π, first observe that all of the spaces to the right of the left-most bar-active space are also bar-active, and that the subsequence σ enclosed by the bar-active spaces avoids 21, $2\overline{1}$, $\overline{21}$, and $1\overline{2}$. Hence σ consists of a sequence of barred entries, in increasing order, followed by a sequence of unbarred entries, in increasing order. We consider three cases.

Case 1: σ has no barred entries.

First observe that when we insert $\overline{n}$ into the left-most space in σ, all spaces in the resulting signed permutation are bar-active. Now observe that when we insert $\overline{n}$ into any other space in σ, we create a pattern of type $1\overline{2}$ consisting of $\overline{n}$ and the necessarily unbarred entry immediately to its left. It follows from these observations that the bar-active spaces in the new signed permutation are those to the right of $\overline{n}$ and the space immediately to the left. Therefore the children of π obtained by inserting $\overline{n}$ have $3, 4, \ldots, f_9(\pi), f_9(\pi)+1, f_9(\pi)+1$ children.

Case 2: σ has no unbarred entries.

First observe that when we insert $\overline{n}$ into the right-most space in σ, the spaces immediately left and right of $\overline{n}$ are bar-active. Moreover, since no bar-active space in π has an unbarred entry to its right and the only forbidden pattern which ends with its second largest entry barred is $\overline{3}1\overline{2}$, all spaces which are bar-active in π are bar-active in the new signed permutation.

Now suppose we insert $\overline{n}$ into a bar-active space other than the right-most space in σ. In this case we create a pattern of type $\overline{21}$ consisting of $\overline{n}$ and the entry immediately to its right. Therefore the bar-active spaces in the new signed permutation are those to the right of $\overline{n}$. It follows that the children of π obtained by inserting $\overline{n}$ have $f_9(\pi)+1, 3, 4, \ldots, f_9(\pi), f_9(\pi)+1$ children.

Case 3: σ has both barred and unbarred entries.

Observe that when we insert $\overline{n}$ into the bar-active space between the last barred entry and the first unbarred entry in σ we create no subsequence of type $\overline{21}$ or $1\overline{2}$ in σ, so the resulting signed permutation has $f_9(\pi)+1$ children. Now suppose we insert $\overline{n}$ among the unbarred entries of σ. Then $\overline{n}$, together with the unbarred entry immediately to its left, forms a $1\overline{2}$ pattern. It follows that the bar-active spaces in the new signed permutation are those to the right of $\overline{n}$ and the space immediately to the left of $\overline{n}$. Finally, suppose we insert $\overline{n}$ among the barred entries of σ. Then $\overline{n}$, together with the barred entry immediately to its right, forms a $\overline{21}$ pattern. It follows that the bar-active spaces in the new signed permutation are those to the right of $\overline{n}$. Combining these observations, we find that the children of π obtained by inserting $\overline{n}$ have $3, 4, \ldots, k+2, f_9(\pi)+1, k+3, \ldots, f_9(\pi), f_9(\pi)+1$ children, where k is the number of unbarred entries in σ.

In all cases the children of π have $3, 4, \ldots, f_9(\pi), f_9(\pi)+1, f_9(\pi)+1$ children, and the result follows. □

Next we describe an isomorphism between the generating tree for $B(T_{10})$ and the classical Schröder generating tree. In contrast to our previous three isomorphisms, this map does not include a detailed description of the locations of the bar-active spaces in a given permutation. Nevertheless, the proof of the theorem restricts these locations somewhat.

Theorem 4.6. *For each signed permutation $\pi\in B(T_{10})$, let $f_{10}(\pi)$ denote one plus the number of bar-active spaces in π. Then the map $\pi\mapsto f_{10}(\pi)$ is an isomorphism of generating trees between the generating tree for $B(T_{10})$ and the classical Schröder generating tree. In particular*,

$$|B_n(21, 2\overline{1}, \overline{231}, \overline{3}1\overline{2})| = r_n \quad (n\geqslant 0).$$

Proof. To begin, observe that $f_{10}(\emptyset)=2$ and $f_{10}(1)=f_{10}(\overline{1})=3$, so both trees have the same first two levels. In view of Lemma 4.1, the quantity $f_{10}(\pi)$ is the number of children of π, so it is sufficient to verify these children have the correct labels.

Fix $n\geqslant 2$ and suppose $\pi\in B_{n-1}(T_{10})$. We consider two cases.

Case 1: π ends with a (possibly empty) sequence σ_u of unbarred entries, which is immediately preceded by a (maximal) nonempty sequence σ_b of barred entries.

We first claim that no space to the left of σ_b which is not adjacent an entry of σ_b is bar-active. This claim is vacuously true if there is no entry to the left of σ_b, so suppose a is the right-most entry left of σ_b. To prove the claim, first observe that a and the barred entry to its right form a sequence of type $2\overline{1}$ or $1\overline{2}$. The first case is forbidden. If we insert $\overline{n}$ to the left of the right-most unbarred entry in the second case we produce a sequence of type $\overline{3}1\overline{2}$, which is forbidden, and the claim follows.

Suppose we insert $\overline{n}$ into a bar-active space in σ_u, so that $\overline{n}$ has an unbarred entry to its left. (Lemma 4.2(i) guarantees such a space exists in σ_u.) The only forbidden pattern of length 3 which has $\overline{2}$ and $\overline{3}$ adjacent is $\overline{231}$ and no entry to the right of $\overline{n}$ in the new signed permutation is barred, so the spaces immediately left and right of $\overline{n}$ are bar-active in the new signed permutation. Since $\overline{3}1\overline{2}$ is forbidden, all other spaces to the left of $\overline{n}$ are bar-inactive in the new signed permutation. Since no forbidden pattern of length 3 has 1, $\overline{2}$, and $\overline{3}$ with $\overline{2}$ and $\overline{3}$ in increasing order, all spaces to the right of $\overline{n}$ which were bar-active in π are bar-active in the new signed permutation.

Now suppose we insert $\overline{n}$ in the space between σ_b and σ_u. (Observe that this space is always bar-active.) As above, the spaces immediately left and right of $\overline{n}$ are bar-active in the new signed permutation. Moreover, if a space was bar-active in π then it is also bar-active in the new signed permutation, since inserting $\overline{n+1}$ in σ_b cannot create a subsequence of type $\overline{3}1\overline{2}$ and inserting $\overline{n+1}$ in σ_u cannot create a subsequence of type $\overline{231}$.

Finally, suppose we insert $\overline{n}$ in a bar-active space in σ_b, so that $\overline{n}$ has a barred entry to its right. As above, the space immediately left of $\overline{n}$ is bar-active, but the space immediately right of $\overline{n}$ is not, since inserting $\overline{n+1}$ in that space produces a $\overline{231}$ pattern, which is forbidden. Similarly, every space to the right of $\overline{n}$ in σ_b is bar-inactive in the new signed permutation. All other spaces which were bar-active in π remain bar-active in the new signed permutation, since inserting $\overline{n+1}$ to the left of $\overline{n}$ cannot create a subsequence of type $\overline{3}1\overline{2}$ and inserting $\overline{n+1}$ in σ_u cannot create a subsequence of type $\overline{231}$.

Combining the observations above, and using the fact that the space between σ_b and σ_u is always bar-active, we find the children of π have $3, 4, \ldots, k+2, f_{10}(\pi)+1, f_{10}(\pi), \ldots, k+3, f_{10}(\pi)+1$ children, where k is the number of bar-active spaces in σ_u.

Case 2: All entries in π are unbarred.

First observe that since π avoids 21, we have $\pi = 12\cdots n-1$, and every space is bar-active. Now suppose we insert $\overline{n}$. As in Case 1, the spaces immediately left and right of $\overline{n}$ are bar-active in the new signed permutation, the rest of the spaces to the left of $\overline{n}$ are bar-inactive, and all spaces to the right of $\overline{n}$ are bar-active. Therefore the children of π have $3, 4, \ldots, f_{10}(\pi), f_{10}(\pi)+1, f_{10}(\pi)+1$ children.

Observe that in both cases the children of π have $3, 4, \ldots, f_{10}(\pi), f_{10}(\pi)+1, f_{10}(\pi)+1$ children, and the result follows. □

We conclude this section by describing an isomorphism between the generating tree for $B(T_{11})$ and the classical Schröder generating tree.

Theorem 4.7. *For each signed permutation $\pi \in B(T_{11})$, let $f_{11}(\pi)$ denote one plus the number of bar-active spaces in π. Then the map $\pi \mapsto f_{11}(\pi)$ is an isomorphism of generating trees between the generating tree for $B(T_{11})$ and the classical Schröder generating tree. In particular,*

$$|B_n(21, 2\overline{1}, \overline{132}, 1\overline{32})| = r_n \quad (n \geqslant 0).$$

Proof. To begin, observe that $f_{11}(\emptyset) = 2$ and $f_{11}(1) = f_{11}(\overline{1}) = 3$, so both trees have the same first two levels. In view of Lemma 4.1, the quantity $f_{11}(\pi)$ is the number of children of π, so it is sufficient to verify these children have the correct labels.

Fix $n \geqslant 2$ and suppose $\pi \in B_{n-1}(T_{11})$. First observe that since no forbidden pattern begins with $\overline{2}$ or $\overline{3}$, the left-most space in π is bar-active. Suppose we insert $\overline{n}$ into the left-most space of π. Since $\overline{2}$ does not appear before $\overline{3}$ in any forbidden pattern, the space immediately right of $\overline{n}$ in the new signed permutation is bar-active. Since no forbidden pattern begins with $\overline{2}$, all spaces which were bar-active in π remain so.

Now suppose we insert $\overline{n}$ in a bar-active space of π other than the left-most space. Since $1\overline{32}$ and $\overline{132}$ are both forbidden, the only bar-active space left of $\overline{n}$ in the new signed permutation is the left-most space. But since $\overline{2}$ does not appear before $\overline{3}$ in any forbidden pattern, the space immediately right of $\overline{n}$ is bar-active, as are all spaces to the right of $\overline{n}$ which were bar-active in π.

Combining the above observations, we find that the children of π have $3, 4, \ldots, f_{11}(\pi), f_{11}(\pi)+1, f_{11}(\pi)+1$ children, and the result follows. □

5. An isomorphism to a new Schröder tree

In this section we enumerate $B_n(T_{12})$, where $T_{12} = \{21, \overline{2}1, \overline{321}, \overline{312}\}$ is as given in the Introduction. As in the previous section, we do this by studying the associated generating tree. In order to describe our results, we need the following new generating tree.

Definition 5.1. The *tilted Schröder tree* is the generating tree given by

- Root: (2);
- Rules: $(2) \to (2)(4)$
  $(k) \to (2)(4)(5)\cdots(k)(k+1)(k+1)$ for $k \geqslant 4$.

In particular, $(4) \to (2)(4)(5)(5)$.

A preliminary examination of the tilted Schröder tree suggests it has r_n nodes on level n. We prove this next.

Theorem 5.2. *For all $n \geqslant 0$, the tilted Schröder generating tree has exactly r_n nodes on level n.*

Proof. We use the kernel method. (See [1,10] for more information on the kernel method.) For any node v in the tilted Schröder tree, let level(v) denote the level of v and let label(v) denote the label of v. Now let $G(x, y)$ be given by

$$G(x,y) = \sum_v x^{\mathrm{level}(v)} y^{\mathrm{label}(v)},$$

where the sum on the right is over the nodes in the tilted Schröder tree. For all $k \geqslant 2$ define $G_k(x)$ by writing

$$G(x,y) = \sum_{k \geqslant 2} G_k(x) y^k. \tag{7}$$

We wish to obtain $G(x, 1)$. To this end, count the children of each node in the tilted Schröder tree to obtain

$$G(x,y) = y^2 + G_2(x)x(y^2+y^4) + x\sum_{k\geqslant 3} G_k(x)(y^2+y^4+y^5+\cdots+y^k+2y^{k+1}). \tag{8}$$

Since every node has exactly one child with label 2, and the root has label 2, we also have

$$G_2(x) = 1 + xG(x,1). \tag{9}$$

Use (9) to eliminate $G_2(x)$ in (8) and use (7) to simplify the result and obtain

$$\left(1 - \frac{xy^2}{y-1} - xy\right) G(x,y) = y^2 + x(y^2+y^4) + \frac{xy^2 - xy^4}{y-1} + G(x,1)\left(x^2(y^2+y^4) - \frac{x^2y^4 + (1-x)xy^2}{y-1} - x^2y^3 - xy^3(1-x)\right).$$

Now set

$$y = \frac{1 + x - \sqrt{1-6x+x^2}}{4x}$$

and solve the resulting equation for $G(x,1)$ to obtain

$$G(x,1)=\frac{1-x-\sqrt{x^2-6x+1}}{2x}.$$

Now the result follows from (1). □

Remark. The author thanks an anonymous referee for pointing out that Theorem 5.2 can also be proved by using [5, Theorem 3.2] with $b=r=c=1$, where P and $f_P(z)$ are the production matrix and the generating function associated with the small Schröder numbers. Alternatively, Julian West has found an 'adoption' procedure which converts the tilted Schröder tree to the classical Schröder tree without changing the number of nodes on any level.

To show that the generating tree for $B(T_{12})$ is isomorphic to the tilted Schröder tree, we study the bar-active and unbar-active spaces in a signed permutation which avoids T_{12}. We begin by characterizing the bar-active spaces.

Lemma 5.3. *Fix $n \geqslant 2$ and suppose $\pi \in B_{n-1}(T_{12})$. Then the following hold.*

(i) *If $\pi(n-1)$ is unbarred then the right-most space in π is bar-active and all other spaces in π are bar-inactive.*
(ii) *If $\pi(n-1)$ is barred then the right-most two spaces in π are bar-active and all other spaces in π are bar-inactive.*

Proof. (i) Since no forbidden pattern ends with its largest entry, the right-most space in π is bar-active. However, if we insert $\overline{n}$ in any space other than the right-most, then $\overline{n}$ and $\pi(n-1)$ form a subsequence of type $\overline{2}1$, which is forbidden.

(ii) Since no forbidden pattern ends with its largest entry, the right-most space in π is bar-active. Similarly, since no forbidden pattern ends with its largest two entries barred, the second space from the right in π is bar-active. Now suppose we insert $\overline{n}$ in a space which is not one of the right-most two spaces in π. If $\pi(n-2)$ is barred then $\overline{n}$, $\pi(n-2)$, and $\pi(n-1)$ form a subsequence of type $\overline{321}$ or $\overline{312}$, both of which are forbidden. If $\pi(n-2)$ is unbarred then $\overline{n}$ and $\pi(n-2)$ form a subsequence of type $\overline{2}1$, which is also forbidden. □

Next we characterize the unbar-active spaces.

Lemma 5.4. *Fix $n \geqslant 2$ and suppose $\pi \in B_{n-1}(T_{12})$. Let σ denote the (possibly empty) sequence of barred entries at the right end of π. Then the following hold.*

(i) *If σ is empty then the right-most space in π is unbar-active and all other spaces in π are unbar-inactive.*
(ii) *Suppose σ is nonempty. Then a space in π is unbar-active if and only if it is adjacent to at least one entry of σ.*

Proof. (i) Since no forbidden pattern ends with its largest entry, the right-most space in π is unbar-active. However, if we insert n in any space other than the right-most, then n and $\pi(n-1)$ form a subsequence of type 21, which is forbidden.

(ii) This is immediate, since the only forbidden subsequence whose largest entry is unbarred is 21. □

We conclude the paper by using Lemmas 5.3 and 5.4 to give an isomorphism between the generating tree for $B(T_{12})$ and the tilted Schröder tree.

Theorem 5.5. *For each signed permutation $\pi \in B(T_{12})$, let $g(\pi)=2$ if π is empty, let $g(\pi)=2$ if the right-most entry of π is unbarred, and let $g(\pi)$ denote three plus the number of barred entries at the right end of π otherwise. Then the map $\pi \mapsto g(\pi)$ is an isomorphism of generating trees between the generating tree for $B(T_{12})$ and the tilted Schröder generating tree. In particular,*

$$|B_n(21,\overline{2}1,\overline{321},\overline{312})| = r_n \quad (n \geqslant 0).$$

Proof. To begin, observe that $g(\emptyset)=2$, $g(1)=2$, and $g(\overline{1})=4$, so both trees have the same first two levels, and it is sufficient to show they have the same succession rules.

Fix $n \geqslant 2$ and suppose $\pi \in B_{n-1}(T_{12})$. We consider two cases.

Case 1: $\pi(n-1)$ is unbarred, so that $g(\pi) = 2$.

In view of Lemmas 5.3 and 5.4, the signed permutation π has $2 = g(\pi)$ children, one of which is obtained by inserting n at the right end of π and the other of which is obtained by inserting $\overline{n}$ at the right end of π. By the same Lemmas, the child obtained by inserting n has 2 children and the child obtained by inserting $\overline{n}$ has 4 children.

Case 2: $\pi(n-1)$ is barred, so that $g(\pi) \geqslant 4$.

In view of Lemmas 5.3 and 5.4, the signed permutation π has $g(\pi)$ children, two of which are obtained by inserting $\overline{n}$ and $g(\pi) - 2$ of which are obtained by inserting n. By the same Lemmas, both children obtained by inserting $\overline{n}$ have $g(\pi) + 1$ children, and the children obtained by inserting n have $2, 4, 5, \ldots, g(\pi)$ children.

These results correspond with the succession rules for the tilted Schröder generating tree, and the result follows from Theorem 5.2. $\square$

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